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RESONANCE FOR SINGULAR PERTURBATION PROBLEMS*

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Abstract. Consider the resonance for a second-order equation $\varepsilon y''-xpy'+qy=0$. Another proof is given for the necessity of the Matkowsky condition and the connection with a regular eigenvalue problem is established. Also, if p, q are analytic, necessary and sufficient conditions are derived.

1. Introduction. Consider the differential equation

$$\varepsilon y''-xp(x,\varepsilon)y'+q(x,\varepsilon)y=0, \qquad -c\leqq x\leqq c. \tag{1.1}$$

Here ε with $0<\varepsilon<1$ is a small constant, $p(x,\varepsilon)\geqq p_0>0$ and $q(x,\varepsilon)$ are smooth functions of x, ε; i.e., $p\in C^\infty(x,\varepsilon)$, $q(x,\varepsilon)\in C^\infty(x,\varepsilon)$. Many authors, for example [2], [6], [11], [12] have studied the asymptotic behavior of the solutions of (1.1) when $\varepsilon\to 0$. In particular Pearson [10] and Ackerberg and O'Malley [1] proved the following basic result. If

$$\frac{q(0,0)}{p(0,0)}\neq 0,1,2,\cdots, \tag{1.2}$$

then for any δ, c with $0<\delta<c$

$$\lim_{\varepsilon\to 0}\frac{\|y\|_{-c+\delta,c-\delta}}{\|y\|_{-c,c}}=0, \qquad \|y\|_{a,b}=\max_{a\leqq x\leqq b}|y(x)|. \tag{1.3}$$

Another proof was given in [5] using the maximum principle.

If (1.2) does not hold, then resonance can occur; i.e., for some constants $c>0$ and $\delta>0$ there is a sequence of solutions with

$$\lim_{\varepsilon\to 0}\frac{\|y\|_{-c+\delta,c-\delta}}{\|y\|_{-c,c}}\neq 0, \qquad 0<\delta<c. \tag{1.4}$$

B. Matkowsky [8] has proposed a sequence of conditions which must be satisfied for resonance to occur. N. Kopell [3] has shown that these conditions are necessary but not sufficient. By changing $q(x,\varepsilon)$ by a quantity which is smaller than any power of ε, resonance can be enforced. Simular results are obtained by F. Olver [9].

In this paper we want to extend the methods of [5] to prove the same results. Also, we shall show that resonance occurs if and only if $\lambda=0$ is an eigenvalue of an associated regular eigenvalue problem. Furthermore, if the coefficients of $p(x,\varepsilon)$, $q(x,\varepsilon)$ are analytic functions of x and smooth functions of ε, then resonance occurs if and only if there is a sequence of analytic solutions

$$y(x,\varepsilon)=\sum_{\nu=0}^{\infty}a_\nu(\varepsilon)x^\nu, \qquad |a_\nu|\leqq K\zeta^\nu, \tag{1.5}$$

K, ζ constants independent of ε, which in a neighborhood of $x=0$ converges uniformly to a nontrivial solution of the reduced equation.

2. Necessary conditions for resonance. In this section we want to prove that the Matkowsky conditions are necessary for resonance. We start with a number of lemmata which are slight generalizations of results in [5].

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LEMMA 2.1. *Consider*

$$\varepsilon y'' - xp(x,\varepsilon)y' + q(x,\varepsilon)y = F(x,\varepsilon), \qquad F(x,\varepsilon) \in C^\infty(x,\varepsilon), \tag{2.1}$$

in an interval $0 < a \leqq x \leqq b \leqq c$. *There are constants* K_0, $\beta > 0$ *which depend only on a and bounds for xp and q such that for all sufficiently small* $\varepsilon > 0$ *and all* δ *with* $0 < \delta \leqq \frac{1}{2}(b-a)$

$$\|y\|_{a,b-\delta} \leqq K_0(\|F\|_{a,b} + |y(a)| + e^{-\beta\delta/\varepsilon}|y(b)|). \tag{2.2}$$

Also, there are constants K_j *which depend only on a and bounds for* $d^\nu(xp)/dx^\nu$, $d^\nu q/dx^\nu$, $\nu = 0, 1, 2, \cdots, j$ *such that*

$$\left\|\frac{d^jy}{dx^j}\right\|_{a,b-\delta} \leqq K_j(1+\varepsilon^{-j}e^{-\beta\delta/\varepsilon})\left(\sum_{\nu=0}^{j}\left\|\frac{d^\nu F}{dx^\nu}\right\|_{a,b} + |y(a)| + \varepsilon^{-j}e^{-\beta\delta/\varepsilon}|y(b)|\right). \tag{2.3}$$

The same estimates hold if $-c \leqq b \leqq x \leqq a < 0$.

Proof. The estimates follow from standard results for singular perturbation problems (see, for example, [7], [11]). □

We shall now show that we can estimate the derivatives d^jy/dx^j in terms of y in intervals which contain the point $x = 0$. These estimates are the main tool of this section.

THEOREM 2.1. *There are constants* C_j *such that for all sufficiently small* ε

$$\left\|\frac{d^jy}{dx^j}\right\|_{-c+\delta,c-\delta} \leqq C_j(1+\varepsilon^{-j}e^{-\beta\delta/\varepsilon})\left(\sum_{\nu=0}^{j}\left\|\frac{d^\nu F}{dx^\nu}\right\|_{-c,c} + \|y\|_{-c,c}\right).$$

Proof. (For more details see [5].) Differentiate (2.1) with respect to x and let

$$v_j = \frac{d^jy}{dx^j}, \qquad F_j = \frac{d^jF}{dx^j}.$$

A simple calculation shows that

$$\varepsilon v_j'' - xpv_j' + (q - j(p + xp'))v_j = F_j + \sum_{\nu=0}^{j-1} A_{j\nu}v_\nu, \tag{2.4}$$

where

$$A_{j\nu} = \binom{j}{\nu-1}(xp)_{j+1-\nu} - \binom{j}{\nu}q_{j-\nu}.$$

By Lemma 2.1 the above estimate is certainly valid for $|x| \geqq \alpha$, where α is any constant with $0 < \alpha \leqq \frac{1}{2}c$. For sufficiently large j, $q - j(p + xp') < 0$ in a neighbourhood of $x = 0$. Here we can estimate v_j in terms of v_ν, $\nu = 0, 1, 2, \cdots, j-1$ using the maximum principle. Therefore we can estimate v_j also in terms of $v_0 = y$. This proves the theorem. □

We need also

LEMMA 2.2. *Let*

$$\frac{q(0,0)}{p(0,0)} = l, \qquad l = 0, 1, 2, \cdots. \tag{2.5}$$

The equation

$$x\phi' - \frac{q(x,0)}{p(x,0)}\phi = g(x) = \sum_{\nu=0}^{l} g_\nu \frac{x^\nu}{\nu!} + x^{l+1}h(x), \qquad h \in C^\infty(x),$$

has a solution belonging to $C^\infty(x)$ *if and only if*

$$g_l = \left.\frac{d^l g}{dx^l}\right|_{x=0} = 0.$$

Proof. We can write

$$\frac{q(x,0)}{p(x,0)} = l + xq_1(x).$$

Let $\phi = x^l\psi$. Then ψ is the solution of

$$\psi' - xq_1\psi = x^{-(l+1)}g(x).$$

ψ is of the form

$$\psi = \sum_{\nu=-l}^{-1} c_\nu x^\nu + c_0 \log|x| + \psi_1, \qquad \psi_1 \in C^\infty(x),$$

and $c_0 = 0$ if and only if $g_l = 0$. This proves the lemma. □

We now can prove that the Matkowsky conditions are necessary. Let (2.5) be satisfied and let $y(x,\varepsilon)$, $\|y(x,\varepsilon)\|_{-c,c} = 1$ be a sequence of solutions of (1.1) which in the interior of $-c < x < c$ converges to a nontrivial solution $\bar{y}$ of the reduced equation

$$xp(x,0)\bar{y}' + q(x,0)\bar{y} = 0.$$

In particular, let x_0 with $x_0 \neq 0$, $|x_0| < c - \delta$ be a point and let ϕ_0 be the solution of

$$-xp(x,0)\phi_0' + q(x,0)\phi_0 = 0, \qquad \phi_0(x_0) = y(x_0,\varepsilon).$$

Then $\lim \phi_0 = \bar{y}$, and $y_1 = y - \phi_0$ satisfies

$$\varepsilon y_1'' - xp(x,\varepsilon)y_1' + q(x,\varepsilon)y_1 = \varepsilon g_1(x,\varepsilon), \qquad y_1(x_0,\varepsilon) = 0. \tag{2.6}$$

y_1 is bounded and $g_1(x,\varepsilon)$ is a smooth function of x, ε. Let

$$\tilde{y}_1 = \frac{y_1}{a}, \qquad a = \max(\varepsilon, \|y_1\|_{-c+\delta/2,c-\delta/2}).$$

Then $\tilde{y}_1$ is the solution of

$$\varepsilon\tilde{y}_1'' - xp(x,\varepsilon)\tilde{y}_1' + q(x,\varepsilon)\tilde{y}_1 = \left(\frac{\varepsilon}{a}\right)g_1(x,\varepsilon), \qquad \tilde{y}_1(x_0,\varepsilon) = 0.$$

$\tilde{y}_1$ is smooth for $|x| \leqq c - \delta$. We want to show that there is a constant τ such that $0 < \tau \leqq \varepsilon/a \leqq 1$. Assume there is no such τ. Then we can assume that $\lim_{\varepsilon\to 0} \varepsilon/a = 0$. Using Theorem 2.1 we can also assume that

$$\lim_{\varepsilon\to 0} \tilde{y}_1(x,\varepsilon) = \psi(x) \quad \text{for } |x| \leqq c - \delta$$

where

$$-xp(x,0)\psi' + q\psi = 0, \qquad \psi(x_0) = 0, \quad \text{i.e.,} \quad \psi(x) \equiv 0.$$

In particular,

$$\lim_{\varepsilon\to 0} \tilde{y}_1(-c+\delta,\varepsilon) = \lim_{\varepsilon\to 0} \tilde{y}_1(c-\delta,\varepsilon) = 0.$$

By Lemma 2.1, applied to the intervals $-c \leqq x \leqq -c+\delta$ and $c - \delta \leqq x \leqq c$ respectively, it follows that $\lim_{\varepsilon\to 0} \|y_1\|_{-c+\delta/2,c+\delta/2} = 0$, which is impossible. Thus we can introduce into (2.6) the new variable $\tilde{\tilde{y}}_1 = y_1/\varepsilon$. Then $\tilde{\tilde{y}}_1$ is bounded for $|x| \leqq c - \frac{1}{2}\delta$ and satisfies

$$\varepsilon \tilde{\tilde{y}}_1'' - xp\tilde{\tilde{y}}_1' + q\tilde{\tilde{y}}_1 = g(x, \varepsilon). \tag{2.7}$$

By Theorem 2.1, $\tilde{\tilde{y}}_1$ converges to a smooth solution of

$$xp(x, 0)\phi_1' + q(x, 0)\phi_1 = g(x, 0),$$

which does not contain any log terms. This is the first of the Matkowsky conditions. We can now repeat the above process and obtain

THEOREM 2.2. *The Matkowsky conditions are necessary for resonance.*

3. An estimate. Consider a system of differential equations

$$dy/dx = A(x)y = F(x). \tag{3.1}$$

Here $F = (F^{(1)}, \cdots, F^{(n)})'$ is a smooth vector function, and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is a smooth complex $n \times n$ matrix.

DEFINITION 3.1. We say that the matrix A is *negative dominant* if there is a constant $\tau > 0$ such that for all $i = 1, 2, \cdots, n$ and all x

$$\text{Real } a_{ii} < 0, \qquad \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \leqq (1 - \tau)|\text{Real } a_{ii}|.$$

In [4, Lemma 2.3] we have proved

LEMMA 3.1. *Assume that A is negative dominant and let*

$$\Lambda(x) = \begin{pmatrix} a_{11}(x) & 0 & \cdots & 0 \\ 0 & a_{22}(x) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn}(x) \end{pmatrix}.$$

Then the solutions of (3.1) *satisfy the estimate*

$$|y(x)| \leqq \tau^{-1} \max_{0 \leqq \eta \leqq x} |\Lambda^{-1}(\eta)F(\eta)| + s(x)|y(0)|, \qquad x \geqq 0, \tag{3.2}$$

where

$$s(x) = \exp\left[\tau \int_0^x a(\xi)\, d\xi\right], \qquad a(x) = \max_i \text{Real } a_{ii}(x).$$

We want to use Lemma 3.1 to estimate the solutions of (2.1). We assume that (2.5) holds.
Then

$$q(0, \varepsilon) \geqq -\rho\varepsilon, \qquad \rho = \text{const.} > 0. \tag{3.3}$$

We write (2.1) in the form

$$\varepsilon y'' - (xpy)' + q_1 y = F, \qquad q_1 = q + (xp)'. \tag{3.4}$$

Let

$$v' = q_1 y - F.$$

Then we can integrate (3.4) to obtain

$$\varepsilon y' - xpy + v = 0,$$

which gives us

$$\begin{pmatrix} y \\ v \end{pmatrix}' = \begin{pmatrix} \dfrac{xp}{\varepsilon} & -\dfrac{1}{\varepsilon} \\ q_1 & 0 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} - \begin{pmatrix} 0 \\ F \end{pmatrix}. \tag{3.5}$$

We want to transform (3.5) into negative dominant form. Introduce new dependent variables by

$$\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 1 & s(x) \\ 0 & 1 \end{pmatrix} u, \quad u = \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}, \quad s(x) = (xp)^{-1}.$$

Then u satisfies

$$u' = \left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} xp/\varepsilon & -1/\varepsilon \\ q_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & s' \\ 0 & 0 \end{pmatrix} \right) u + \begin{pmatrix} sF \\ -F \end{pmatrix}$$

$$= \begin{pmatrix} xp/\varepsilon - q_1/xp & (xp)^{-2}((xp)' - q_1) \\ q_1 & q_1/xp \end{pmatrix} u + \begin{pmatrix} F/xp \\ -F \end{pmatrix}.$$

Let $\alpha > 0$, $\beta > 0$ be constants which we choose later. Then

$$w = e^{-\beta x} \begin{pmatrix} \alpha x & 0 \\ 0 & 0 \end{pmatrix} u$$

is the solution of

$$w' = \begin{pmatrix} \dfrac{xp}{\varepsilon} - \beta - \dfrac{1}{x}\left(\dfrac{q_1}{p} + 1\right) & \dfrac{\alpha}{xp^2}((xp)' - q_1) \\ \dfrac{q_1}{\alpha x} & \dfrac{q_1}{xp} - \beta \end{pmatrix} w + \begin{pmatrix} \dfrac{F}{\alpha p} \\ -F \end{pmatrix}. \tag{3.6}$$

We consider now an interval $-c \leqq x \leqq -\gamma\sqrt{\varepsilon}$, $\gamma > 0$ constant independent of ε, and shall show that we can choose α, β, γ such that the system (3.6) is negative dominant.

1) Choose α such that $|q_1/\alpha x| \leqq \frac{1}{2}|q_1/xp|$.
2) By (4.3), $q_1(0, \varepsilon) \geqq 1 + O(\varepsilon)$, and we can choose β such that $q_1/xp - \beta \leqq -1$ for all x with $-c \leqq x < 0$.
3) Choose γ so large that for $-c \leqq x \leqq -\gamma\sqrt{\varepsilon}$

$$\frac{xp}{\varepsilon} - \beta - \frac{1}{x}\left(\frac{q_1}{p} + 1\right) \leqq -\frac{1}{2}\frac{p_0|x|}{\varepsilon}, \quad \frac{\alpha}{xp^2}((xp)' - q_1) < \frac{1}{4}\frac{p_0|x|}{\varepsilon}.$$

With these choices of α, β, γ the system (3.6) is negative dominant with $\tau = \frac{1}{4}$.

We write the solution of (3.6) in the form

$$w = w_1 + \begin{pmatrix} 0 \\ u_S \end{pmatrix}$$

where u_s is the solution of

$$u_S' = \left(\frac{q_1(x,0)}{xp(x,0)} - \beta\right)u_S - F, \qquad u_S(-1) = 0, \tag{3.7}$$

and w_1 satisfies

$$w_1' = \tilde{A}w_1 + G, \qquad G = (G^{(1)}, G^{(2)})', \tag{3.8}$$

where $\tilde{A}$ is the same matrix as in (3.6) and

$$G^{(1)} = \frac{F}{\alpha p} - \frac{1}{\alpha x p^2}((xp)' - q_1)u_S, \quad G^{(2)} = \frac{\varepsilon}{x}\frac{1}{\varepsilon}\left(\frac{q_1(x,\varepsilon)}{p(x,\varepsilon)} - \frac{q_1(x,0)}{p(x,0)}\right)u_S.$$

The solution of (3.7) can be estimated by

$$|u_S(x)| \leqq K_1 \begin{cases} |x|\,|\log|x||\max\limits_{-c\leqq\xi\leqq x}|F(\xi)| & \text{if } q_1(0,0)/p(0,0) = 1, \\ |x|\max\limits_{-c\leqq\xi\leqq x}|F(\xi)| & \text{if } q_1(0,0)/p(0,0) > 1. \end{cases}$$

The system (3.8) is negative dominant. Therefore, by Lemma 3.1,

$$\begin{aligned} |w_1(x)| &\leqq K_2 \max_{-c\leqq\xi\leqq x}\left(\left|\frac{\varepsilon}{\xi}F(\xi)\right| + \left|\frac{\varepsilon}{\xi^2}u_S(\xi)\right|\right) + |w_1(-c)| \\ &\leqq K_2 \max_{-c\leqq\xi\leqq x}\left(\left|\frac{\sqrt{\varepsilon}}{\gamma}F(\xi)\right| + \left|\frac{1}{\gamma^2}u_S(\xi)\right|\right) + |w_1(-c)|. \end{aligned}$$

We now return to the original variables and obtain

$$\begin{aligned} |y(x)| + |v(x)| &\leqq \frac{e^\beta}{\alpha|x|}(|u_S(x)| + |w_1(x)|) \\ &\leqq K_3|x|^{-1}(|y(-c)| + \varepsilon|y'(-c)|) + K_3\max_{-c\leqq\xi\leqq x}|F(\xi)| \\ &\quad\cdot\begin{cases} |\log|x|| & \text{if } \dfrac{q_1(0,0)}{p(0,0)} = 1, \\ 1 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.9}$$

Here K_j are constants which do not depend on ε.

The corresponding estimate holds in the interval $\gamma\sqrt{\varepsilon} \leqq x \leqq c$.

4. The analytic case. In this section we consider a sequence of solutions $y(x, \varepsilon)$, $\varepsilon \to 0$, of (2.1) which are uniformly bounded in an interval $|x| \leqq a$, i.e.,

$$\|y(x,\varepsilon)\|_{-a,a} \leqq K_0. \tag{4.1}$$

We assume that $p(x, \varepsilon)$, $q(x, \varepsilon)$, $F(x, \varepsilon)$ are smooth functions of ε and analytic functions of x for $x \in \Omega$. Here Ω is an open domain in the complex plane which contains the interval $|x| \leqq a$. We also assume that p, q, F are uniformly bounded for all ε and all $x \in \Omega$. Therefore, by Cauchy's integral formula, there are constants ζ_1, K such that

$$\begin{gathered} \|F_\nu\|_{-a,a} + \|q_\nu\|_{-a,a} + \|(xp)_\nu\|_{-a,a} \leqq K_1\nu!\zeta_1^\nu, \\ F_\nu = d^\nu F/dx^\nu. \end{gathered} \tag{4.2}$$

It is well known that for every fixed ε the solutions of (2.1) are analytic functions of x

for $x \in \Omega$, i.e. we can continue the above solutions analytically into Ω. Let $x=-b$ be a point with $-a<-b<0$. We want to show that the $y(x, \varepsilon)$ are uniformly bounded in a complex neighborhood of $x=-b$.

By (4.1) and Theorem 2.1 there is a constant K such that

$$|y(-a, \varepsilon)|+\varepsilon|y'(-a, \varepsilon)| \leqq K. \tag{4.3}$$

Now rewrite the differential equation (2.1) as the first-order system (3.5) and consider it on the half-lines

$$x=r e^{i\phi}-a, \quad r \geqq 0, \quad -\phi_0 \leqq \phi \leqq \phi_0.$$

There it has the form

$$\frac{d}{dr}\begin{pmatrix} y \\ v \end{pmatrix}=e^{i\phi}\begin{pmatrix} \dfrac{xp}{\varepsilon} & -\dfrac{1}{\varepsilon} \\ q_1 & 0 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}+e^{i\phi}\begin{pmatrix} 0 \\ F \end{pmatrix}.$$

For all sufficiently small ϕ the interval $0 \leqq r \leqq a-\frac{1}{2}b$ belongs to Ω. Also, the estimates of § 3 are still valid. Therefore, by (4.3) and (3.9) the solutions $y(x, \varepsilon)$ are uniformly bounded in a complex neighbourhood of $x=-b$. The same is true for $x=b$. Using Cauchy's integral formula we obtain

$$|v_\nu(-b, \varepsilon)|+|v_\nu(b, \varepsilon)| \leqq K_2 \nu! \zeta_2^\nu, \quad v_\nu=\frac{d^\nu y}{dx^\nu}. \tag{4.4}$$

Without restriction we can assume that $K_2=K_1$, $\zeta_2=\zeta_1$.

Now choose for b the largest number with the properties

$$b \leqq \tfrac{1}{2}a, \quad \min_{|x| \leqq b}(p+xp') \geqq \tfrac{1}{2}p_0, \quad p_0=\min_{|x| \leqq a} p(x).$$

Let j_0 be the smallest positive integer such that $q_0-\frac{1}{2}j_0 p_0<0$, $q_0=\max_{|x| \leqq a} q$. Using the maximum principle we obtain from (2.4) for $j \geqq j_0$

$$\begin{aligned}
\|v_j\|_{-b,b} &\leqq \frac{1}{(1/2)jp_0-q_0}\left(\|F_j\|_{-b,b}+\left\|\sum_{\nu=0}^{j-1} A_{j\nu} v_\nu\right\|_{-b,b}\right)+|v_j(-b, \varepsilon)|+|v_j(b, \varepsilon)| \\
&\leqq K_3 j! \zeta_1^j+\frac{K_1}{(1/2)jp_0-q_0} \sum_{\nu=0}^{j-1}\left(\binom{j}{\nu-1}(j+1-\nu)! \zeta_1^{j+1-\nu}+\binom{j}{\nu}(j-\nu)! \zeta_1^{j-\nu}\right)\|v_\nu\|_{-b,b} \\
&\leqq K_3 j! \zeta_1^j+\frac{K_1 j! \zeta_1^j}{(1/2)jp_0-q_0} \sum_{\nu=0}^{j-1}(\zeta_1 \nu+1) \frac{\|v_\nu\|_{-b,b}}{\zeta_1^\nu \nu!} \\
&\leqq j! \zeta_1^j\left(K_3+K_4 \sum_{\nu=0}^{j-1} \frac{\|v_\nu\|_{-b,b}}{\zeta_1^\nu \nu!}\right).
\end{aligned}$$

Here

$$K_3=K_1\left(1+\frac{1}{(1/2)j_0 p_0-q_0}\right), \quad K_4=K_1 \max_{j \geqq j_0} \frac{(j-1)\zeta_1+1}{(1/2)jp_0-q_0}.$$

Let $\tilde{\alpha}$ be the lower bound of all α satisfying

$$\alpha^j \geqq K_3+K_4 \sum_{\nu=0}^{j-1} \alpha^\nu \quad \text{for } j \geqq j_0, \qquad \alpha^j \geqq \frac{\|v_j\|_{-b,b}}{\zeta_1^j j!} \quad \text{for } j<j_0.$$

Then the above inequality for $\|v_j\|_{-b,b}$ gives us

$$\|v_j\|_{-b,b} \leqq \tilde{\alpha}^j \zeta_1^j j!.$$

Therefore the solutions $y(x, \varepsilon)$ can be expanded into the power series (1.5) with $\zeta = \tilde{\alpha}\zeta_1$ and the result stated in the introduction is proved.

5. The associated eigenvalue problem. We assume in this section that the Matkowsky conditions are satisfied and that $c = 1$. We want to construct the general solution of (1.1) in the interval $-1 \leqq x \leqq -\gamma\sqrt{\varepsilon}$.

Consider the reduced equation

$$-xp(x, 0)u_0' + q(x, 0)u_0 = 0, \qquad \left.\frac{d^l u_0}{dx^l}\right|_{x=0} = 1.$$

By assumption it has a smooth solution of the form

$$u_0(x) = x^l\phi_0(x), \quad \phi_0(x) \geqq C_0 > 0, \quad C_0 = \text{const.} \tag{5.1}$$

Let $y(x)$ be a solution of the full equation (1.1). Then $y_1 = y - u_0$ satisfies

$$\varepsilon y_1'' - xp(x, \varepsilon)y_1' + q(x, \varepsilon)y_1 = \varepsilon(u_0''(x) + F_1(x, \varepsilon)), \tag{5.2}$$

where

$$u_0''(x) = x^l\phi_0'' + 2lx^{l-1}\phi_0' + l(l-1)x^{l-2}\phi_0,$$

$$\varepsilon F_1 = x(p(x, \varepsilon) - p(x, 0))u_0' - (q(x, \varepsilon) - q(x, 0))u_0 = \varepsilon x^l\psi_1(x)$$

are smooth functions of x, ε. By assumption the equation

$$-xp(x, 0)u_1' + q(x, 0)u_1 = u_0''(x) + F_1(x, 0), \qquad \left.\frac{d^l u_1}{dx^l}\right|_{x=0} = 0$$

also has a smooth solution which is of the form

$$u_1(x) = x^{l-2}(a_{10}l(l-1) + a_{11}lx + a_{12}x^3 + \cdots) = x^{l-2}\phi_1(x).$$

Therefore $y_2 = y_1 - u_1 = y - u_0 - \varepsilon u_1$ satisfies

$$\varepsilon y_2'' - xp(x, \varepsilon)y_2' + q(x, \varepsilon)y_2 = \varepsilon^2(u_1'' + F_2(x, \varepsilon)),$$

which is of the same form as (5.2). Thus we can repeat the process and obtain after n steps

$$y = \sum_{\nu=0}^{n-1} \varepsilon^\nu u_\nu + y_n, \tag{5.3}$$

where

$$u_\nu = x^{\tau_\nu}\phi_\nu(x), \quad \phi_\nu(x) \text{ smooth}, \quad \tau_\nu = \max(l - 2\nu, 0), \tag{5.4}$$

and $y_n = y_n(x, \varepsilon)$ is the solution of

$$\varepsilon y_n'' - xp(x, \varepsilon)y_n' + q(x, \varepsilon)y_n = \varepsilon^n F_n(x, \varepsilon), \tag{5.5}$$

with initial values

$$y_n(-1, \varepsilon) = y(-1, \varepsilon) - \sum_{\nu=0}^{n-1} \varepsilon^\nu u_\nu(-1),$$

$$y_n'(-1, \varepsilon) = y'(-1, \varepsilon) - \sum_{\nu=0}^{n-1} \varepsilon^\nu u_\nu'(-1).$$

The above expansion is valid for all solutions of (1.1). In particular we can choose the initial values such that

(5.6) $$y_n(-1, \varepsilon) = y_n'(-1, \varepsilon) = 0.$$

By (3.9)

$$|y_n(x, \varepsilon)| \leqq \text{const. } \varepsilon^n |\log |x||, \quad 0 < \varepsilon \leqq \varepsilon_0, \quad -1 \leqq x \leqq -\gamma\sqrt{\varepsilon}.$$

Also, if we write (5.5) in the form

$$\begin{aligned}\varepsilon y_n'(x, \varepsilon) &= \int_{-1}^{x} \xi p(\xi, \varepsilon) y_n'(\xi, \varepsilon) - q(\xi, \varepsilon) y_n(\xi, \varepsilon) + \varepsilon^n F_n(\xi, \varepsilon)\, d\xi \\ &= xp(x, \varepsilon) y_n(x, \varepsilon) - \int_{-1}^{x} ((\xi p(\xi, \varepsilon))' + q(\xi, \varepsilon)) y_n(\xi, \varepsilon) - \varepsilon^n F_n(\xi, \varepsilon)\, d\xi,\end{aligned}$$

it follows that

$$\left|\frac{\partial y_n(x, \varepsilon)}{\partial x}\right| \leqq \text{const. } \varepsilon^{n-1} |\log |x||,$$

and therefore we obtain from (5.5)

(5.7) $$\left|\frac{\partial^\nu y_n(x, \varepsilon)}{\partial x^\nu}\right| \leqq \text{const. } \varepsilon^{n-\nu} |\log |x||.$$

Differentiating (5.5) and the boundary conditions (5.6) with respect to ε gives us, for $v = \partial y_n/\partial \varepsilon$,

$$\varepsilon v'' - xpv' + qv = \varepsilon^{n-2} \tilde{F},$$

$$v(-1, \varepsilon) = v(-1, \varepsilon) = 0,$$

which is an equation of the same type as (5.5). Therefore

$$|v| = \left|\frac{\partial y_n}{\partial \varepsilon}\right| \leqq \text{const. } \varepsilon^{n-2} |\log |x||,$$

and by (5.7)

$$\left|\frac{\partial^{\nu+1} y_n}{\partial x^\nu\, \partial \varepsilon}\right| \leqq \text{const. } \varepsilon^{n-\nu-2} |\log |x||.$$

In general we have

(5.8) $$\left|\frac{\partial^{\nu_1+\nu_2} y_n}{\partial x^{\nu_1}\, \partial \varepsilon^{\nu_2}}\right| \leqq \text{const. } \varepsilon^{n-\nu_1-2\nu_2} |\log |x||.$$

The above process can be applied for any n, ν_1, ν_2. Therefore, by choosing n sufficiently large, we can construct a solution

$$w_n(x, \varepsilon) = \sum_{\nu=0}^{n-1} \varepsilon^\nu u_\nu(x) + y_n(x, \varepsilon),$$

which for $-1 \leqq x \leqq -\gamma\sqrt{\varepsilon}$ has any (but fixed) number of derivatives bounded independently of ε.

Remark. The important point of (5.8) is that we can estimate the derivatives up to x-values of order $O(-\sqrt{\varepsilon})$. In any other interval $-1 < b \leqq x \leqq a < 0$, a, b independent of ε, these estimates follow already from (2.3) for all bounded solutions of (1.1), provided $y(-1, \varepsilon)$ and $y'(-1, \varepsilon)$ are smooth functions of ε.

By (5.4) and (5.1) we have, for $-1 \leqq x \leqq -\gamma\sqrt{\varepsilon}$,

$$|\varepsilon^{\nu} u_\nu| \leqq \text{const.} \begin{cases} |x|^l \left(\dfrac{\varepsilon^{1/2}}{x}\right)^{\nu} & \text{for } \nu \leqq l/2 \\ |\varepsilon|^{\nu} & \text{for } \nu > l/2. \end{cases}$$

Therefore, by (5.1), there are constants $c_j > 0$ such that for $2n \geqq l$, sufficiently large γ and sufficiently small ε

$$c_1|x|^l \leqq |w_n(x, \varepsilon)| \leqq c_2|x|^l, \quad -1 \leqq x \leqq -\gamma\sqrt{\varepsilon}. \tag{5.9}$$

We determine now the general solution of (1.1) in the usual way. Let $w_n(x, \varepsilon)$ be the above solution. Then all other solutions satisfy

$$v'w_n - vw_n' = \left(\varepsilon^{-1} \exp \varepsilon^{-1} \int_{-1}^{x} \xi p(\xi, \varepsilon)\, d\xi\right).$$

An easy calculation shows that

$$v_n(x, \varepsilon) = -\varepsilon^{-1} w_n(x, \varepsilon) \int_x^{-\gamma\sqrt{\varepsilon}} (w_n(\eta, \varepsilon))^{-1} \exp\left(\varepsilon^{-1} \int_{-1}^{\eta} \xi p(\xi, \varepsilon)\, d\xi\right) d\eta$$

is another linearly independent solution. By (5.9) it has the properties

$$\begin{aligned} v_n(-1, \varepsilon) &= \frac{-1}{p(-1, 0)} + O(\varepsilon), \\ |v_n(x, \varepsilon)| &\leqq \text{const.}\, \frac{x+1}{\varepsilon} \exp\left(-\frac{1}{\varepsilon} \int_{-1}^{x} \xi p(\xi, \varepsilon)\, d\xi\right). \end{aligned} \tag{5.10}$$

$v_n(x, \varepsilon)$ and all its derivatives are exponentially small outside the boundary layer at $x = -1$.

All bounded solutions of (1.1) which are not exponentially small outside the boundary layer can be written as

$$y(x, \varepsilon) = \rho_1(w_n(x, \varepsilon) + \rho_2 v_n(x, \varepsilon)), \quad \rho_1, \rho_2 \text{ bounded}. \tag{5.11}$$

Therefore at $x_0 = -\gamma\sqrt{\varepsilon}$

$$\begin{aligned} \sqrt{\varepsilon}\, \frac{y'(x_0, \varepsilon)}{y(x_0, \varepsilon)} &= \sqrt{\varepsilon}\, \frac{w_n'(x_0, \varepsilon)}{w_n(x_0, \varepsilon)} + O(e^{-\alpha/\varepsilon}) \\ &= a_0(\varepsilon) + O(e^{-\alpha/\varepsilon}), \qquad \alpha \sim \left|\int_{-1}^{0} xp(x, \varepsilon)\, dx\right|. \end{aligned} \tag{5.12}$$

For $a_0(\varepsilon)$ we obtain the asymptotic expansion

$$\begin{aligned} \frac{\sqrt{\varepsilon}\, w_n'(x_0, \varepsilon)}{w_n(x_0, \varepsilon)} &= a_0(\varepsilon) \\ &= \sqrt{\varepsilon} \sum_{\nu=0}^{n-1} \varepsilon^{\nu} u_\nu'(x_0) \Big/ \sum_{\nu=0}^{n-1} \varepsilon^{\nu} u_\nu(x_0) + O(\varepsilon^{n-(l+1)/2} |\log \varepsilon|). \end{aligned} \tag{5.13}$$

For $\gamma\sqrt{\varepsilon} \leqq x \leqq 1$ we can proceed in the same way. Thus all bounded solutions which away from the boundary layer are not exponentially small are of the form

$$\tilde{y}(x, \varepsilon) = \rho_1(\tilde{w}_n(x, \varepsilon) + \rho_2 \tilde{v}_n(x, \varepsilon)), \qquad \rho_1, \rho_2 \text{ bounded},$$

where

$$\tilde{w}_n(x, \varepsilon) = \sum_{\nu=0}^{n-1} \varepsilon^\nu u_\nu(x) + \tilde{y}_n(x, \varepsilon), \qquad |\tilde{y}_n| \leqq \text{const. } \varepsilon^n |\log \varepsilon|.$$

(Observe that the $u_\nu(x)$ are defined for all $|x| \leqq 1$.) At $x_1 = \gamma\sqrt{\varepsilon}$ these solutions satisfy the relation

$$(5.14) \qquad \begin{aligned} \sqrt{\varepsilon}\, \tilde{y}'(x_1, \varepsilon)/\tilde{y}(x_1, \varepsilon) &= \sqrt{\varepsilon}\, \tilde{w}_n(x_1, \varepsilon)/\tilde{w}_n(x_1, \varepsilon) + O(e^{-\beta/\varepsilon}) \\ &= a_1(\varepsilon) + O(e^{-\beta/\varepsilon}), \qquad \beta \sim \left| \int_0^1 xp(x, \varepsilon)\, dx \right| \end{aligned}$$

where

$$a_1(\varepsilon) = \frac{\sqrt{\varepsilon} \sum_{\nu=0}^{n-1} \varepsilon^\nu u'_\nu(x_1)}{\sum_{\nu=0}^{n-1} \varepsilon^\nu u_\nu(x_1)} + O(\varepsilon^{n-(l+1)/2} |\log \varepsilon|).$$

The associated eigenvalue problem is given by

$$(5.15) \qquad \begin{aligned} &\varepsilon\phi'' - xp(x, \varepsilon)\phi' + q(x, \varepsilon)\phi = \lambda\phi, \qquad -\gamma\sqrt{\varepsilon} \leqq x \leqq \gamma\sqrt{\varepsilon}, \\ &\sqrt{\varepsilon}\, \phi'(x_j) - a_j\phi(x_j) = 0, \qquad j = 0, 1, \end{aligned}$$

which can be written as the regular eigenvalue problem

$$(5.16) \qquad \begin{aligned} &\ddot{\psi} - zp(\sqrt{\varepsilon}\, z, \varepsilon)\dot{\psi} + q(\sqrt{\varepsilon}\, z, \varepsilon)\psi = \lambda\psi, \qquad -\gamma \leqq z \leqq \gamma, \\ &\dot{\psi}(-\gamma) - a_0\psi(-\gamma) = 0, \quad \dot{\psi}(\gamma) - a_1\psi(\gamma) = 0, \quad \dot{\psi} = \frac{d\psi}{dz}, \end{aligned}$$

by introducing $x = \sqrt{\varepsilon} \cdot z$ as a new variable.

THEOREM 5.1. *Resonance can only occur if* (5.16) *has an eigenvalue* λ *of order* $O(e^{-\gamma/\varepsilon})$, $\gamma = \text{const.} > 0$. *If* $\lambda = 0$ *is an eigenvalue of* (5.16) *for a sequence* $\varepsilon \to 0$, *then resonance occurs. If the Matkowsky conditions are satisfied* (5.16) *has an eigenvalue* λ *with*

$$(5.17) \qquad |\lambda| \leqq \text{const. } |\varepsilon^m|, \quad \textit{for any positive integer } m.$$

Therefore we can find a function $\tilde{q}(z, \varepsilon) \in C^\infty$ *with* $\tilde{q}(z, \varepsilon) = 0$ *for* $|z| \geqq \frac{1}{2}\gamma$ *and* $\tilde{q}(z, \varepsilon) > 0$ *otherwise such that for the modified problem*

$$\varepsilon y'' - xp(x, \varepsilon)y' + (q(x, \varepsilon) + \varepsilon^m \tilde{q}(x/\sqrt{\varepsilon}, \varepsilon))y = 0$$

resonance occurs.

Proof. If resonance occurs then there is a sequence of solutions of (1.1),

$$y(x, \varepsilon) = \rho_1(w_n(x, \varepsilon) + \rho_2 v_n(x, \varepsilon)), \quad \rho_1 > \delta > 0, \quad \rho_2 \text{ bounded}.$$

These solutions satisfy

$$\begin{aligned} &\varepsilon y'' - xp(x, \varepsilon)y' + q(x, \varepsilon)y = 0, \qquad x_0 = -\gamma\sqrt{\varepsilon} \leqq x \leqq \gamma\sqrt{\varepsilon} = x_1, \\ &\sqrt{\varepsilon}\, y'(x_j) - a_j y(x_j) = O(e^{-\gamma/\varepsilon}). \end{aligned}$$

Therefore, by Lemmata A1 and A3 of the Appendix, (5.15) and (5.16) must have an exponentially small eigenvalue.

Assume now that $\lambda = 0$ is an eigenvalue of (5.16) and (5.15) for a sequence $\varepsilon \to 0$. Let $\phi(x, \varepsilon)$ be the corresponding eigenfunction. Then

$$y(x, \varepsilon) = \begin{cases} w_n(x, \varepsilon) & \text{for } x \leqq -\gamma\sqrt{\varepsilon}, \\ \dfrac{w_n(x_0, \varepsilon)}{\phi(x_0, \varepsilon)}\phi(x, \varepsilon) & \text{for } |x| \leqq \gamma\sqrt{\varepsilon}, \\ \dfrac{\phi(x_1, \varepsilon)}{\phi(x_0, \varepsilon)} \cdot \dfrac{w_n(x_0, \varepsilon)}{\tilde{w}_n(x_1, \varepsilon)} \tilde{w}_n(x, \varepsilon) & \text{for } x \geqq \gamma\sqrt{\varepsilon}, \end{cases}$$

denotes a sequence of solutions of (1.1) which do not converge to zero.

If the Matkowsky conditions are satisfied then by (5.3), (5.13) and (5.14) the function

$$s(x) = \varepsilon^{-l/2} \sum_{\nu=0}^{n-1} u_\nu(x)$$

satisfies

$$\varepsilon s'' - xp(x, \varepsilon)s' + q(x, \varepsilon)s = O(\varepsilon^{n-(l/2)}),$$
$$\sqrt{\varepsilon}\, s'(x_j) - a_j s(x_j) = O(\varepsilon^{n-(l+1)/2}|\log \varepsilon|).$$

Here n is arbitrary and therefore (5.17) follows from Lemmata A1 and A3. The resonance of the modified problem follows from Lemmata A2 and A3. (Observe that $\tilde{q}(x/\sqrt{\varepsilon}, \varepsilon)$ does not affect w_n, $\tilde{w}_n$ because $\tilde{q} = 0$ for $|x| \geqq \gamma\sqrt{\varepsilon}$.) □

Appendix. In this section we collect some facts about eigenvalue problems. We consider selfadjoint equations

$$\text{(A1)} \qquad \begin{aligned} L[y] = y'' + f(x)y = \lambda y, \quad 0 \leqq x \leqq 1, \quad f(x) \in C^\infty, \\ y'(0) + \alpha y(0) = 0, \qquad y'(1) + \beta y(1) = 0, \end{aligned}$$

and denote by

$$(u, v) = \int_0^1 \bar{u}v\, dx, \qquad (u, u) = \|u\|^2$$

the usual L_2-scalar product and norm. The following lemma is well known.

LEMMA A1. *Let $u(x)$ with $\|u\| = 1$ be a function satisfying*

$$u'' + f(x)u = G(x),$$
$$u'(0) + \alpha u(0) = g_0, \qquad u'(1) + \beta u(1) = g_1.$$

Then (A1) *has an eigenvalue λ and a corresponding eigenfunction φ with*

$$|\lambda| + \|\varphi - u\| \leqq \text{const. } (\|G\| + |g_0| + |g_1|).$$

We need also

LEMMA A2. *Assume that* (A1) *has an eigenvalue λ_0 with $|\lambda_0| \ll 1$ and let φ_0 denote the corresponding eigenfunction. Let $g(x)$ be a smooth function with*

$$(\varphi_0(x), g(x)\varphi_0(x)) \neq 0 \qquad (\text{e.g., } g(x) \geqq 0,\ g(x) \not\equiv 0).$$

If $|\lambda_0|$ is sufficiently small then we can find a σ with

$$|\sigma| \leqq \text{const. } |\lambda_0|$$

such that $\lambda = 0$ *is an eigenvalue of the perturbed eigenvalue problem*

$$\text{(A2)}\qquad \begin{gathered} \tilde{L}[w] = w'' + (f(x) + \sigma g(x))w = \lambda w, \\ w'(0) + \alpha w(0) = 0, \qquad w'(1) + \beta w(1) = 0. \end{gathered}$$

Proof. Let λ_j with $|\lambda_0| < |\lambda_1| < |\lambda_2| \cdots$ and φ_j denote the eigenvalues and eigenfunctions of (A1) respectively. We want to construct the desired solution in the form

$$w = \varphi_0 + \sum_{j=1}^{\infty} \hat{w}_j \varphi_j$$

i.e.

$$\text{(A3)}\qquad 0 = \tilde{L}[w] = \lambda_0 \varphi_0 + \sigma g \varphi_0 + \sum_{j=1}^{\infty} \lambda_j \hat{w}_j \varphi_j + \sigma q \sum_{j=1}^{\infty} \hat{w}_j \varphi_j.$$

Multiplying (A3) by φ_ν and forming the scalar product gives us the equivalent system

$$\text{(A4)}\qquad \begin{gathered} \lambda_0 + \sigma(\varphi_0, g\varphi_0) + \sigma \sum_{j=1}^{\infty} \hat{w}_j (\varphi_0, g\varphi_0) = 0, \\ \lambda_\nu \hat{w}_\nu + \sigma \sum_{j=1}^{\infty} \hat{w}_j (\varphi_\nu, g\varphi_j) = -\sigma(\varphi_\nu, g\varphi_0). \end{gathered}$$

Let $\tilde{w}_\nu = \lambda_\nu \hat{w}_\nu$; then (A4) becomes

$$\text{(A5)}\qquad \lambda_0 + \sigma(\varphi_0, g\varphi_0) + \sigma \sum_{j=1}^{\infty} \frac{\tilde{w}_j}{\lambda_j} (\varphi_0, g\varphi_j) = 0,$$

$$\text{(A6)}\qquad \tilde{w}_\nu + \sigma \sum_{j=1}^{\infty} \frac{\tilde{w}_j}{\lambda_j} (\varphi_\nu, g\varphi_j) = -\sigma(\varphi_\nu, g\varphi_0).$$

Observing that $|\lambda_j^{-1}| \leqq \text{const.}/j^2$ for $j \geqq 1$ it follows that for sufficiently small $|\sigma|$ the system (A6) has a solution $\tilde{w}_\nu \sim -\sigma(\varphi_\nu, g\varphi_0)$. Therefore, if $|\lambda_0|$ is sufficiently small, (A5) can be solved for $\sigma \sim -\lambda_0/(\varphi_0, g\varphi_0)$. This proves the lemma. □

More general eigenvalue problems

$$\text{(A7)}\qquad \begin{gathered} y'' + h(x)y' + f(x)y = \lambda y, \\ y'(0) + \alpha y(0) = 0, \qquad y'(1) + \beta y(1) = 0 \end{gathered}$$

can be transformed into self-adjoint form and we have

LEMMA A3. *The results of Lemmata* A1 *and* A2 *are also valid for the problems* (A7).

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